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(Nov–Dec)

FYUP : 1st Semester Examination

MAJOR

MATHEMATICS

(Fundamental Mathematics—I)

MTH–100

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **one** question from each Unit

UNIT—I

1. *(a)* Use ε-δ definition to show that

$$\lim_{x\to 2} \frac{2x^2-8}{x-2} = 8$$ 4

(b) A function $f(x)$ is defined as follows :

$$f(x) = 3+2x \text{ ; for } -\tfrac{3}{2} \le x < 0$$
$$= 3-2x \text{ ; for } 0 \le x < \tfrac{3}{2}$$
$$= -3-2x\text{; for } x \ge \tfrac{3}{2}$$

Show that $f(x)$ is continuous at $x = 0$ and discontinuous at $x = \frac{3}{2}$. 4

(c) Evaluate : 3×2=6

(i) $\lim_{x\to\infty}(x-\sqrt{x^2+x})$

(ii) $\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}$

(d) State the intermediate value theorem. Use the theorem to show that the equation $\cos x = x$ has at least one solution. 1+3=4

2. (a) Find $f(0)$ so that $f(x)=\frac{1-\cos x}{x^2}$ for $x \neq 0$ may be continuous at $x=0$. 4

(b) Use ε-δ definition to show that $f(x)$ is continuous at $x=0$

$$f(x) = x\sin(\tfrac{1}{x}), \text{ when } x \neq 0$$
$$= 0 \quad , \text{ when } x = 0$$

Find δ if $\varepsilon = 0\cdot 1$. 3+1=4

(c) Show that the limit

$$\lim_{x\to 2}\frac{|x-2|}{x-2}$$

does not exist. 3

(d) Evaluate : 3

$$\lim_{x\to\infty}\frac{x+\sin x}{x+\cos x}$$

(e) State and prove fixed-point theorem. 1+3=4

UNIT—II

3. (a) Differentiate $\sin x$ w.r.t. x^2. 2

(b) If $y=(\sin^{-1}x)^2$, prove that—

(i) $(1-x^2)y_2 - xy_1 = 2$

(ii) $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n = 0$

2+3=5

(c) Find the range of values for which the function $f(x)=x^3-6x^2-36x+7$ increases with x. 3

(d) Show that the curve $y^3 = 8x^2$ is concave to the foot of the ordinate everywhere except at the origin. 3

(e) Discuss the applicability of the mean value theorem $f(b)-f(a)=(b-a)f(\xi)$, $a<\xi<b$. Find ξ, if the theorem be applicable for the function

$f(x)=x(x-1)(x-3);\ 0\le x\le 4$ 2+4=6

4. (a) If a function has a finite derivative at a point, prove that it is continuous at that point. Show also by an example that the converse is not necessarily true. 3+2=5

(b) Find the nth derivative of

$$y = \frac{1}{ax+b}$$ 3

(c) Find the points of inflexion of the curve $y = e^{-x^2}$. 2

(d) Show that

$$\frac{x}{1+x} < \log(1+x) < x \quad \text{for } x > 0$$ 5

(e) Use differentials to find approximate value of $\cos 61°$. 4

UNIT—III

5. (a) Evaluate by the method of summation, $\int_1^2 (3x^2+2)$ and verify the result with the help of fundamental theorem. 4+1=5

(b) Evaluate : 3×2=6

(i) $\int \frac{2x+1}{x^2+2x+5}\,dx$

(ii) $\int \frac{dx}{(1-x)\sqrt{1-x^2}}$

(c) Obtain a reduction formula for

$$\int_0^{\frac{\pi}{4}} \tan^n x\,dx$$

n being a positive integer greater than 1. 4

(d) Find the area above the x-axis, included between the parabola $y^2 = ax$ and the circle $x^2+y^2 = 2ax$. 4

6. (a) Prove that

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}\,dx = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$$ 5

(b) If $f(x) = |x-2|$, show that $\int_0^3 f(x)\,dx = \frac{5}{2}$. 3

(c) Evaluate : 3

$$\int \sqrt{\frac{x-1}{x+1}}\,dx$$

(d) Find the length of the arc of the parabola $x^2 = 4y$ from the vertex to the point where $x = 2$. 4

(e) Find the volume of the solid formed by rotation of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ about the major axis. 4

UNIT—IV

7. *(a)* If z_1 and z_2 are two complex numbers, then show that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and

$$\text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp}\, z_1 - \text{amp}\, z_2.$$ 3+2=5

(b) Solve the equation $x^3 - 12x + 65 = 0$ by Cardan's method. 5

(c) Express $x^5 + 5x^3 + 3x$ as a polynomial in $(x-1)$. 4

(d) Find the condition that the equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

should have the roots connected by the relation $\beta + \gamma = \alpha + \delta$. 5

8. *(a)* Prove that the equation $x^4 + 15x^2 + 7x - 11 = 0$ has two real roots, one positive and the other negative and two other complex roots. 4

(b) Solve the equation

$$x^4 - 13x^3 + 53x^2 - 83x + 42 = 0$$

if two of the roots are 1 and 2. 5

(c) *(i)* State De Moivre's theorem.

(ii) Solve the equation

$$x^8 + x^5 - x^3 - 1 = 0$$ 1+4=5

(d) Solve the equation

$$6x^3 - 11x^2 + 6x - 1 = 0$$

given that the roots are in AP. 5

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